Gov 2002: Midterm Practice

Spring 2021

Exercises

Problem 1

(a) Is it possible that an event is independent of itself? If so, when?

(b) Is it always true that if A and B are independent events, then A^c and B^c are independent events? Show that it is, or give a counterexample.

(c) Give an example of 3 events A, B, C which are not independent, yet satisfy $P(A \cap B \cap C) = P(A)P(B)P(C)$. Hint: consider simple and extreme cases.

Problem 2

A crime is committed by one of two suspects, A and B. Initially, there is equal evidence against both of them. In further investigation at the crime scene, it is found that the guilty party had a blood type found in 10% of the population. Suspect A does match this blood type, whereas the blood type of Suspect B is unknown.

(a) Given this new information, what is the probability that A is the guilty party?

(b) Given this new information, what is the probability that B's blood type matches that found at the crime scene?

Problem 3

A stick is broken into two pieces, at a uniformly random chosen break point. Find the CDF and average of the length of the longer piece.

Problem 4

For $X \sim \text{Pois}(\lambda)$, find $E(X!)$ (the average factorial of X), if it is finite.

Problem 5

Let U be a Uniform r.v. on the interval $(-1, 1)$ (be careful about minus signs).

(a) Compute $E(U)$, $Var(U)$, and $E(U^4)$.

(b) Find the CDF and PDF of U^2. Is the distribution of U^2 Uniform on $(0, 1)$?

Problem 6

Let $U \sim \text{Unif}(0, 1)$, and

$$X = \log\left(\frac{U}{1-U}\right).$$

Then X has the Logistic distribution.

(a) Write down (but do not compute) an integral giving $E(X^2)$.

(b) Find $E(X)$ without using calculus.

Hint: A useful symmetry property here is that $1 - U$ has the same distribution as U.

Problem 7

Two fair six-sided dice are rolled (one green and one orange), with outcomes X and Y respectively for the green and the orange.

(a) Compute the covariance of $X + Y$ and $X - Y$

(b) Are $X + Y$ and $X - Y$ independent? Show that they are or that they aren't (whichever is true).

Problem 8

Emails arrive in an inbox according to a Poisson process with rate λ so the number of emails in a time interval of length t is distributed $Pois(\lambda t)$ and the numbers of emails arriving in disjoint time intervals are independent. Let X, Y, Z be the numbers of emails that arrive from 9am to noon, noon to 6pm and 6pm to midnight on a certain day.

(a) Find the joint PMF of X, Y, Z

(b) Find the conditional joint PMF of X, Y, Z given that $X + Y + Z = 36$

Solutions

Answer 1

(a) Let A be an event. If A is independent of itself, then $P(A) = P(A \cap A) = P(A)^2$, so $P(A)$ is 0 or 1. So this is only possible in the extreme cases that the event has probability 0 or 1.

(b) Yes, because we have

$$P(A^c\ B^c) = 1P(A \cup B) = 1 - (P(A) + P(B) - P(A\ B))$$

.

since A and B are independent, this becomes

$$1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))(1 - P(B)) = P(A^c)P(B^c)$$

(c) Consider the extreme case where $P(A) = 0$. Then it's automatically true that $P(A)P(B)P(C) = 0$, and also $P(A \cap B \cap C) = 0$ since $A \cap B \cap C$ is a subset of A (and in general, if event A_1 is a subset of event A_2, then $P(A_1) \leq P(A_2)$). Then take B and C to be dependent, e.g., take any example with $B = C$ and $0 < P(B) < 1$.

Answer 2

(a)

Let M be the event that A's blood type matches the guilty party's and for brevity, write A for "A is guilty" and B for "B is guilty". By Bayes' Rule,

$$P(A|M) = \frac{P(M|A)P(A)}{P(M|A)P(A) + P(M|B)P(B)} = \frac{1/2}{1/2 + (1/10)(1/2)} = \frac{10}{11}.$$

(We have $P(M|B) = 1/10$ since, given that B is guilty, the probability that A's blood type matches the guilty party's is the same probability as for the general population.)

(b)

Let C be the event that B's blood type matches, and condition on whether B is guilty. This gives

$$P(C|M) = P(C|M, A)P(A|M) + P(C|M, B)P(B|M) = \frac{1}{10} \cdot \frac{10}{11} + \frac{1}{11} = \frac{2}{11}.$$

Answer 3

We can assume the units are chosen so that the stick has length 1. Let L be the length of the longer piece, and let the break point be $U \sim \text{Unif}(0,1)$. For any $l \in [1/2, 1]$, observe that $L < l$ is equivalent to $U < l, 1-U < l$, which can be written as $1 - l < U < l$. We can thus obtain L's CDF as

$$F_L(l) = P(L < l) = P(1 - l < U < l) = 2l - 1,$$

so $L \sim \text{Unif}(1/2, 1)$ and $E(L) = 3/4$.

Answer 4

By LOTUS,

$$E(X!) = e^{-\lambda}\sum_{k=0}^{\infty} k!\frac{\lambda^k}{k!} = \frac{e^{-\lambda}}{1-\lambda},$$

for $0 < \lambda < 1$ since this is a geometric series (and $E(X!)$ is infinite if $\lambda \geq 1$).

Answer 5

(a)

We have $E(U) = 0$ since the distribution is symmetric about 0. By LOTUS,

$$E(U^2) = \frac{1}{2}\int_{-1}^{1} u^2 du = \frac{1}{3}.$$

So $\text{Var}(U) = E(U^2) - (EU)^2 = E(U^2) = \frac{1}{3}$. Again by LOTUS,

$$E(U^4) = \frac{1}{2}\int_{-1}^{1} u^4 du = \frac{1}{5}.$$

(b)

Let $G(t)$ be the CDF of U^2. Clearly $G(t) = 0$ for $t \leq 0$ and $G(t) = 1$ for $t \geq 1$, because $0 \leq U^2 \leq 1$. For $0 < t < 1$,

$$G(t) = P(U^2 \leq t) = P(-\sqrt{t} \leq U \leq \sqrt{t}) = \sqrt{t},$$

since the probability of U being in an interval in $(-1, 1)$ is proportional to its length. The PDF is $G'(t) = \frac{1}{2}t^{-1/2}$ for $0 < t < 1$ (and 0 otherwise). The distribution of U^2 is *not* Uniform on $(0,1)$ as the PDF is not a constant on this interval.

Answer 6

(a) By LOTUS,

$$E(X^2) = \int_0^1 \left(\log \left(\frac{u}{1-u} \right) \right)^2 du.$$

(b) By the symmetry property mentioned in the hint, $1 - U$ has the same distribution as U. So by linearity,

$$E(X) = E(\log U - \log(1-U)) = E(\log U) - E(\log(1-U)) = 0.$$

Answer 7

(a)

$$\text{Cov}(X+Y, X-Y) = \text{Cov}(X,X) - \text{Cov}(X,Y) + \text{Cov}(Y,X) - \text{Cov}(Y,Y) = 0.$$

(b)

They are not independent: information about $X+Y$ may give information about $X-Y$, as shown by considering an *extreme example*. Note that if $X+Y=12$, then $X=Y=6$, so $X-Y=0$. Therefore, $P(X-Y=0|X+Y=12) = 1 \neq P(X-Y=0)$, which shows that $X+Y$ and $X-Y$ are not independent. Alternatively, note that $X+Y$ and $X-Y$ are both even or both odd, since the difference $X+Y-(X-Y)=2Y$ is even.

Answer 8

(a)

Since $X \sim \text{Pois}(3\lambda), Y \sim \text{Pois}(6\lambda), Z \sim \text{Pois}(6\lambda)$ independently, the joint PMF is

$$P(X=x, Y=y, Z=z) = \frac{e^{-3\lambda}(3\lambda)^x}{x!} \frac{e^{-6\lambda}(6\lambda)^y}{y!} \frac{e^{-6\lambda}(6\lambda)^z}{z!},$$

for any nonnegative integers x, y, z.

(b)

Let $T = X + Y + Z \sim \text{Pois}(15\lambda)$, and suppose that we observe $T = t$. The conditional PMF is 0 for $x + y + z \neq t$. For $x + y + z = t$,

$$\begin{aligned}
P(X = x, Y = y, Z = z | T = t) &= \frac{P(T = t | X = x, Y = y, Z = z) P(X = x, Y = y, Z = z)}{P(T = t)} \\
&= \frac{\frac{e^{-3\lambda}(3\lambda)^x}{x!} \frac{e^{-6\lambda}(6\lambda)^y}{y} \frac{e^{-6\lambda}(6\lambda)^z}{z!}}{\frac{e^{-15\lambda}(15\lambda)^t}{t!}} \\
&= \frac{t!}{x! y! z!} \left(\frac{3}{15}\right)^x \left(\frac{6}{15}\right)^y \left(\frac{6}{15}\right)^z.
\end{aligned}$$