$X_1, \dots, X_n$ be iid $\underline{N}(\mu, \sigma^2)$

$n \geq 2$

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \overline{X}_n)^2$$

$$\Rightarrow \quad \overline{X}_n \perp\!\!\!\perp S_n^2$$

Approach: work w/

$$(\overline{X}_n, \underbrace{X_1 - \overline{X}_n, \dots, X_n - \overline{X}_n}_{\text{deviations}})$$

mean

$$\mathrm{Cov}(\overline{X}_n, X_i - \overline{X}_n)$$

$$= \mathrm{Cov}(\overline{X}_n, X_i) - \mathrm{Cov}(\overline{X}_n, \overline{X}_n)$$

$$= \mathrm{Cov}(\bar{X}_n, X_i) - V[\bar{X}_n]$$

$$V[\bar{X}_n] = \sigma^2/n$$

$$\mathrm{Cov}[\bar{X}_n, X_i] =$$

$$\mathrm{Cov}\left(\frac{1}{n}\sum_{j=1}^{n} X_j, X_i\right)$$

$$= \frac{1}{n}\sum_{j=1}^{n} \mathrm{Cov}(X_j, X_i)$$

if $j \neq i \Rightarrow \mathrm{Cov}(X_j, X_i) = 0$

$$= \frac{1}{n}\mathrm{Cov}(X_i, X_i)$$

$$= \frac{V[X_i]}{n} = \frac{\sigma^2}{n}$$

$$\mathrm{Cov}(\bar{X}_n, X_i - \bar{X}_n) = 0$$

$$\bar{X}_n \perp\!\!\!\perp \{X_1 - \bar{X}_n, \ldots, X_n - \bar{X}_n]$$

$$\Rightarrow \bar{X}_n \perp\!\!\!\perp S_n^2$$