Example: CEF as a function of binary X.

$\mu(x) = E[Y|X=x]$ $\qquad X \in \{0, 1\}$

$$E[Y|X=1] = \mu(1) = \mu_1$$

$$E[Y|X=0] = \mu(0) = \mu_0$$

$$E[Y|X=x] = \begin{cases} \mu_1 & \text{if } x=1 \\ \mu_0 & \text{if } x=0 \end{cases}$$

$$\underline{\mu(x) = x\mu_1 + (1-x)\mu_0}$$

$$= \mu_0 + x(\mu_1 - \mu_0)$$

$$\alpha + \beta x$$

Stick of length 1 broken @ $X \sim \text{Unif}(0,1)$. Given $X=x$, another break at $Y \sim \text{Unif}(0,x)$.

$$E[Y|X]$$

$$\mu(x) = E[Y|X=x]$$

$$= x/2$$

$$E[Y|X] = \mu(X) = \underline{X/2}$$

$$E[E[Y|X]] = E[X/2]$$

$$= \frac{1}{2} E[X] = \frac{1}{4}$$

<u>LIE proof</u>

$$E[E[Y|X]]$$

$$= \sum_x \underline{E[Y|X=x]}\, Pr[X=x]$$

$$= \sum_x \sum_y y\, Pr[Y=y|X=x]\, Pr[X=x]$$

$$\boxed{\underline{Pr[Y=y|X=x]} = \frac{Pr[Y=y, X=x]}{\underline{Pr[X=x]}}}$$

$$= \sum_x \sum_y y\, Pr[Y=y, X=x]$$

$$= E[Y]$$

Survey weights

$R=1$ if polled

$\underline{X=x}$ $\quad$ $R=0$ if not

Inclusion prob.

$$\pi(x) = \Pr[R=1 \mid X=x]$$

$E[Y] \Rightarrow$ goal

$E[Y \mid R=1]$

Random sampling w/i $X=x$ strata implies

$$Y \perp\!\!\!\perp R \mid X$$

$\Rightarrow$ create inverse prob. weight (IPW)

$$\Rightarrow \quad \frac{YR}{\pi(X)} \quad \text{(Horvitz Thompson estimator)}$$

$$E\left[\frac{YR}{\pi(X)}\right] = E\left[E\left[\frac{YR}{\pi(X)} \,\middle|\, X\right]\right]$$

$$= \sum_x E\left[\frac{YR}{\pi(x)} \,\middle|\, X=x\right] Pr[X=x]$$

$$= \sum_x \frac{1}{\pi(x)} E[YR \mid X=x]\, Pr[X=x]$$

$$= \sum_x \frac{1}{\pi(x)} \Big(E[Y \cdot 1 \mid R=1, X=x]\, Pr[R=1 \mid X=x] + E[Y \cdot 0 \mid R=0, X=x]\, Pr[R=0 \mid X=x]\Big) \times Pr[X=x]$$

$$= \sum_x \frac{1}{\cancel{\pi(x)}} E[Y \mid R=1, X=x] \, \cancel{\Pr[R=1 \mid X=x]} \times \Pr[X=x]$$

$$= \sum_x E[Y \mid R=1, X=x] \Pr[X=x]$$

$$= \sum_x E[Y \mid X=x] \Pr[X=x]$$

$$= E\big[E[Y \mid X] \big]$$

$$= E[Y]$$