7. Normal Distribution

Spring 2021

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Gov 2002 (Harvard)

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- Learning about r.v. distributions in all combinations
    - Discrete vs. continuous and joint vs. marginal
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- But first: the normal distribution and its cousins.
    - Why: massively important to inference due to the central limit theorem.

Standard normal distribution

Definition

A continuous r.v. Z follows a **standard normal distribution** if its p.d.f. φ is given as

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad -\infty < z < \infty,$$

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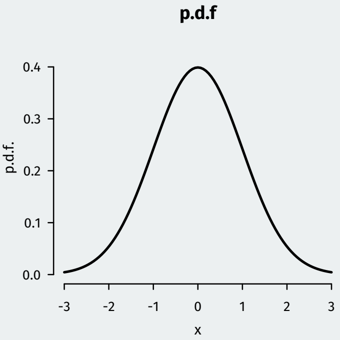
- Standard normal is mean zero, variance 1: $\mathbb{E}[Z] = 0, \mathbb{V}[Z] = 1$.

The normal distribution



- Deeply symmetric:

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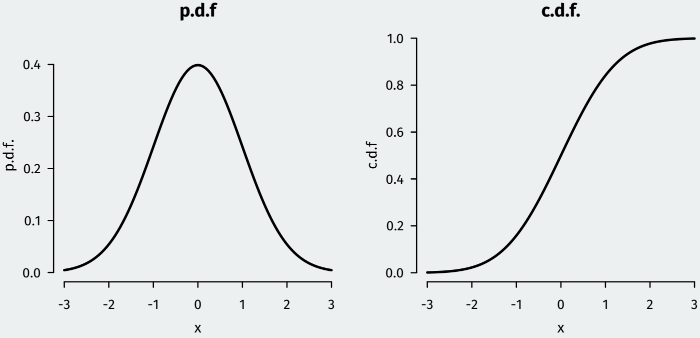

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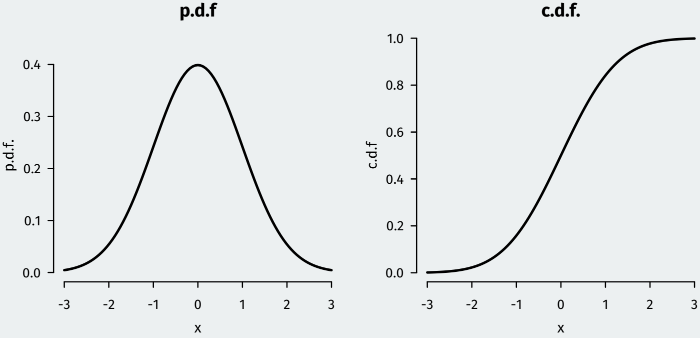

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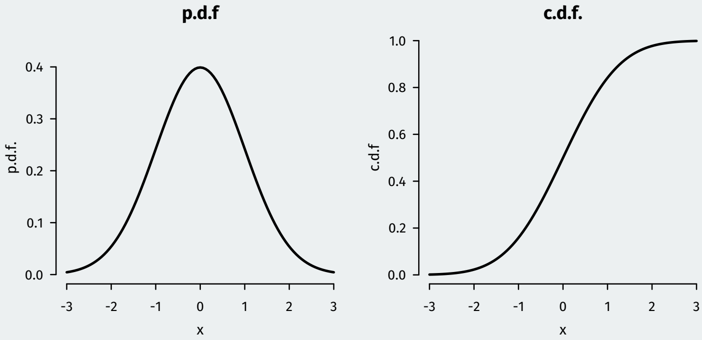

- Deeply symmetric:
  - p.d.f. is symmetric: $\varphi(z) = \varphi(-z)$
  - Tail areas are symmetric $\Phi(z) = 1 - \Phi(-z)$
  - Z and $-Z$ are both $\mathcal{N}(0, 1)$

General normal distribution

Defintion

If $Z \sim \mathcal{N}(0,1)$ then

$$X = \mu + \sigma Z$$

follows the normal distribution with mean μ and variance σ^2, written $X \sim \mathcal{N}(\mu, \sigma^2)$.

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- p.d.f.:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

Properties of normals and sums

- If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ and $X_1 \perp\!\!\!\perp X_2$,

$$X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

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$$X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

- **Cramer's theorem:** if $X_1 \perp\!\!\!\perp X_2$ and $X_1 + X_2$ is normal, then X_1 and X_2 are normal.

Using pnorm

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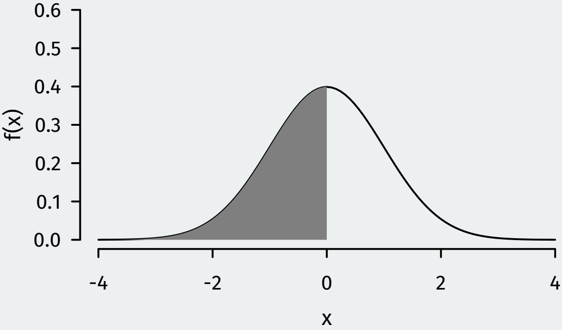


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pnorm(q = 0, mean = 0, sd = 1)
```

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## [1] 0.5
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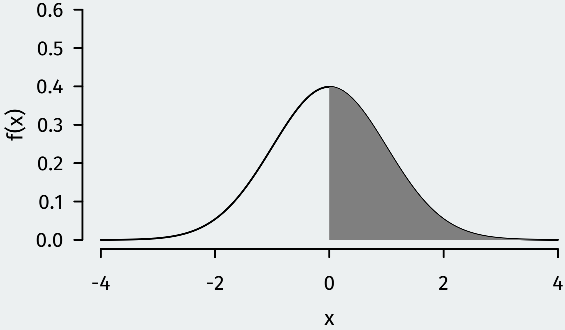


```r
pnorm(q = 0, mean = 0, sd = 1, lower.tail = FALSE)
```

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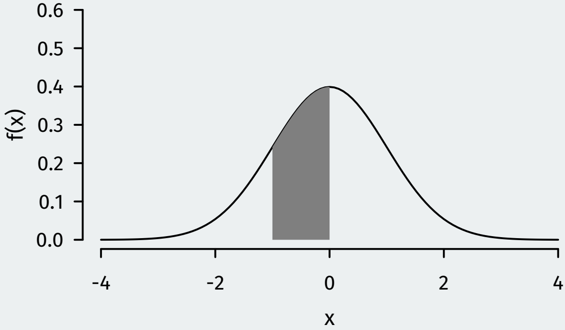
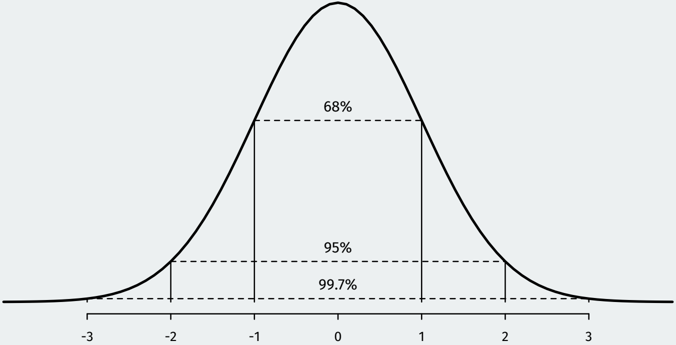



```r
pnorm(q = 0, mean = 0, sd = 1) - pnorm(q = -1, mean = 0, sd = 1)
```

```
## [1] 0.341
```

Empirical Rule for the Normal Distribution



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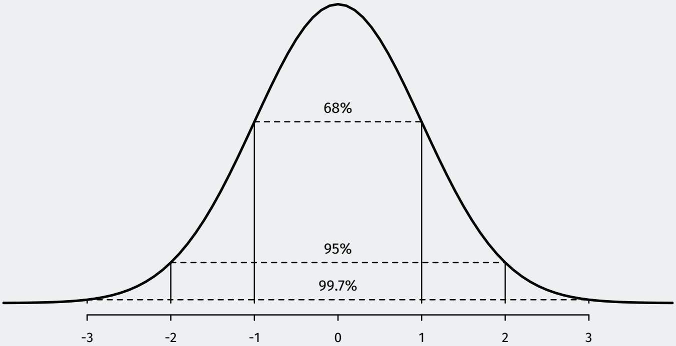

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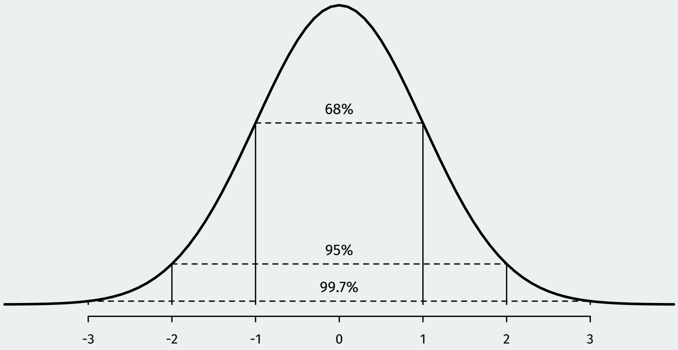

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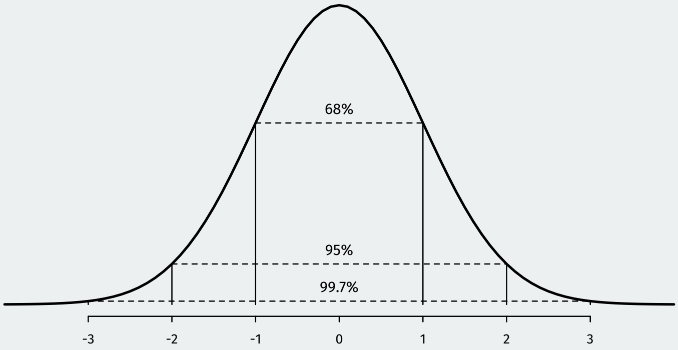

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  - Roughly 68% of the distribution of Z is between -1 and 1.
  - Roughly 95% of the distribution of Z is between -2 and 2.
  - Roughly 99.7% of the distribution of Z is between -3 and 3.

Chi-square distribution

Definition

Let $V = Z_1^2 + \cdots + Z_n^2$ where $Z_1, Z_2, \ldots, Z_n$ are i.i.d. $\mathcal{N}(0,1)$. Then V follows the **Chi-square distribution** with n degrees of freedom, written $V \sim \chi_n^2$

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- Why do we care? **Sample variance** of normal r.v.s $X_1, \ldots, X_n$ i.i.d. $N(\mu, \sigma^2)$:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2 \qquad \frac{(n-1)s^2}{\sigma^2} \sim \chi_{n-1}^2$$

Student t distribution

Definition

If $Z \sim \mathcal{N}(0,1)$ and $V \sim \chi_n^2$ with $Z \perp\!\!\!\perp V$, then

$$T = \frac{Z}{\sqrt{V/n}},$$

follows the **student-t distribution** with n degrees of freedom, written $T \sim t_n$.

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- Important result for the **normal model**: if $X_1, \dots, X_n$ are i.i.d. $\mathcal{N}(\mu, \sigma^2)$:

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  - Fatter tails than the normal.
  - Converges to $\mathcal{N}(0,1)$ as $n \to \infty$

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- Expectation of a random vector is just the vector of expectations:

$$\mathbb{E}[\mathbf{X}] = (\mathbb{E}[X_1], \mathbb{E}[X_2], \ldots, \mathbb{E}[X_k])'$$

Covariance matrices

- Covariance matrix generalizes (co)variance to this setting:

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$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \cdots & \sigma_k^2 \end{pmatrix}$$

where, $\sigma_j^2 = \mathbb{V}[X_j]$ and $\sigma_{ij} = \text{Cov}(X_i, X_j)$.

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- Symmetric ($\mathbf{\Sigma} = \mathbf{\Sigma}'$) because $\text{Cov}(X_i, X_j) = \text{Cov}(X_j, X_i)$.

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    - $\mathbf{I}_k$ is the k by k identity matrix because $\mathbb{V}[Z_j] = 1$ and $\text{Cov}(Z_i, Z_j) = 0$.

Linear transformations of random vectors

Theorem

If $\mathbf{X} \in \mathbb{R}^k$ with $k \times 1$ expectation $\boldsymbol{\mu}$ and $k \times k$ covariance matrix $\boldsymbol{\Sigma}$, and $\mathbf{A}$ is a $q \times k$ matrix, then $\mathbf{AX}$ is a random vector with mean $\mathbf{A}\boldsymbol{\mu}$ and covariance matrix $\mathbf{A\Sigma A}'$.

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  - $\boldsymbol{\mu}$: $q \times 1$ mean vector $\mathbb{E}[\mathbf{X}] = \boldsymbol{\mu}$
  - $\boldsymbol{\Sigma} = \mathbf{BB}'$: $q \times q$ covariance matrix $\mathbb{V}[\mathbf{X}] = \boldsymbol{\Sigma}$.

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- Let $\mathbf{Z} \sim \mathcal{N}(0, \mathbf{I}_k)$ and $\mathbf{X} = \boldsymbol{\mu} + \mathbf{BZ}$, where $\mathbf{B}$ is $q \times k$ then $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
  - $\boldsymbol{\mu}$: $q \times 1$ mean vector $\mathbb{E}[\mathbf{X}] = \boldsymbol{\mu}$
  - $\boldsymbol{\Sigma} = \mathbf{BB}'$: $q \times q$ covariance matrix $\mathbb{V}[\mathbf{X}] = \boldsymbol{\Sigma}$.
- More generally, if $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ then $\mathbf{Y} = \mathbf{a} + \mathbf{BX} \sim \mathcal{N}(\mathbf{a} + \mathbf{B}\boldsymbol{\mu}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}')$

Properties of the multivariate normal

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- If (X, Y) are multivariate normal with $\text{Cov}(X, Y) = 0$, then X and Y are independent.