9. Asymptotics

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Matthew Blackwell

Gov 2002 (Harvard)

Where are we? Where are we going?

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- Last time: introducing estimators, looking at finite-sample properties.
- Now: can we say more as sample size grows?

1/ Asymptotics

Current knowledge

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- For i.i.d. r.v.s, $X_1, \ldots, X_n$, with $\mathbb{E}[X_i] = \mu$ and $\mathbb{V}[X_i] = \sigma^2$ we know that:
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    - What if the data isn't normal? What is the sampline distribution of $\overline{X}_n$?
- **Asymptotics**: approximate the sampling distribution of $\overline{X}_n$ as n gets big.

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- Note: this is a sequence of random variables!

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A sequence $\{a_n : n = 1, 2, ...\}$ has the **limit** a written $a_n \to a$ as $n \to \infty$ if for all $\delta > 0$ there is some $n_\delta < \infty$ such that for all $n \geq n_\delta$, $|a_n - a| \leq \delta$.

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- $\{a_n : n = 1, 2, ...\}$ is **bounded** if there is $b < \infty$ such that $|a_n| < b$ for all n.

Convergence in Probability

Definition

A sequence of random variables, $\{Z_n : n = 1, 2, ...\}$, is said to **converge in probability** to a value b if for every $\varepsilon > 0$,

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    - Distribution of $\hat{\theta}_n$ collapses on θ as $n \to \infty$.
    - Inconsistent estimator are bad bad bad: more data gives worse answers!

Chebyshev Inequality

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Suppose that X is r.v. for which $\mathbb{V}[X] < \infty$. Then, for every real number $\delta > 0$,

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq \delta) \leq \frac{\mathbb{V}[X]}{\delta^2}.$$

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- Variance places limits on how far an observation can be from its mean.

Proof of Chebyshev

- Let $Z = X - \mathbb{E}[X]$ with density $f_Z(x)$. Probability is just integral over the region:

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- Note that where $|x| \geq \delta$, we have $1 \leq x^2/\delta^2$, so

$$\mathbb{P}\left(|Z| \geq \delta\right) \leq \int_{|x| \geq \delta} \frac{x^2}{\delta^2} f_Z(x) dx \leq \int_{-\infty}^{\infty} \frac{x^2}{\delta^2} f_Z(x) dx = \frac{\mathbb{E}[Z^2]}{\delta^2} = \frac{\mathbb{V}[X]}{\delta^2}$$

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- Proof similar to Chebyshev.

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- NB: Unbiasedness does not imply consistency, nor vice versa.

Law of large numbers

Weak Law of Large Numbers

Let $X_1, \ldots, X_n$ be a an i.i.d. draws from a distribution with mean $\mathbb{E}[X_i] < \infty$. Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, $\overline{X}_n \xrightarrow{p} \mathbb{E}[X_i]$.

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- Intuition: The probability of $\overline{X}_n$ being "far away" from μ goes to 0 as n gets big.
- Implies many sample means converge:
    - If $\mathbb{E}[X_i^2] < \infty$, then $\frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{p} \mathbb{E}[X_i^2]$

LLN by simulation in R

- Draw different sample sizes from Exponential distribution with rate 0.5

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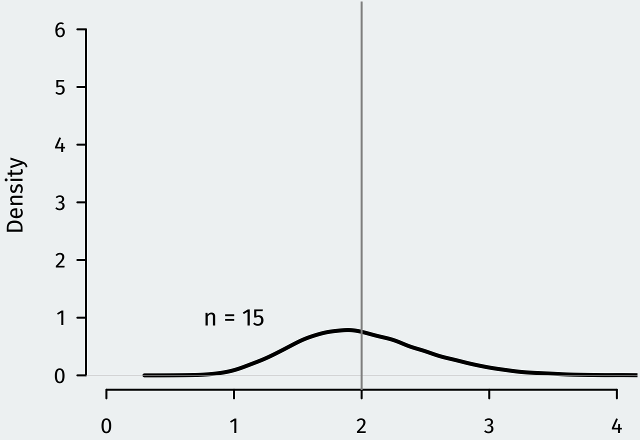
```
nsims <- 10000
holder <- matrix(NA, nrow = nsims, ncol = 6)
for (i in 1:nsims) {
  s5 <- rexp(n = 5, rate = 0.5)
  s15 <- rexp(n = 15, rate = 0.5)
  s30 <- rexp(n = 30, rate = 0.5)
  s100 <- rexp(n = 100, rate = 0.5)
  s1000 <- rexp(n = 1000, rate = 0.5)
  s10000 <- rexp(n = 10000, rate = 0.5)

  holder[i,1] <- mean(s5)
  holder[i,2] <- mean(s15)
  holder[i,3] <- mean(s30)
  holder[i,4] <- mean(s100)
  holder[i,5] <- mean(s1000)
  holder[i,6] <- mean(s10000)
}
```

LLN in action



- Distribution of $\overline{X}_{15}$

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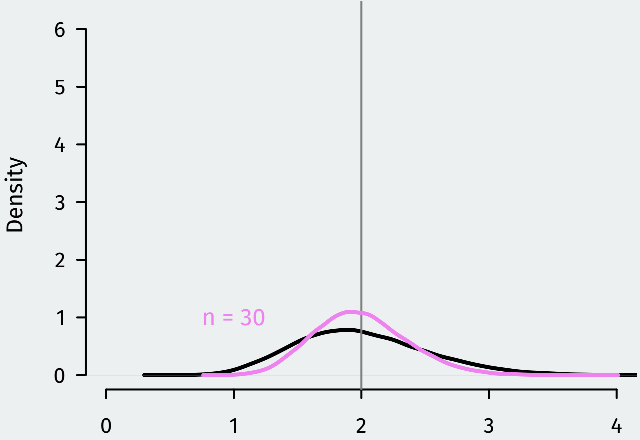
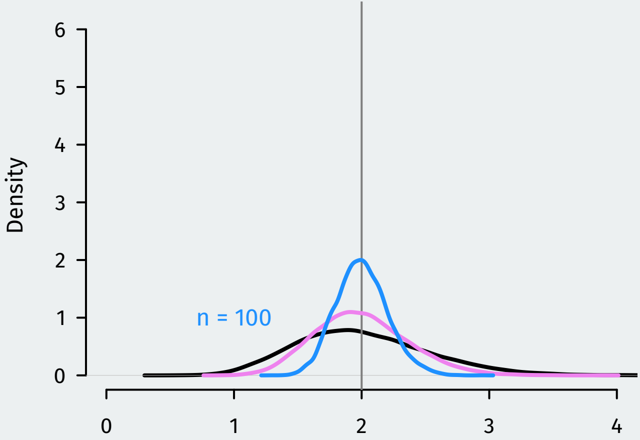


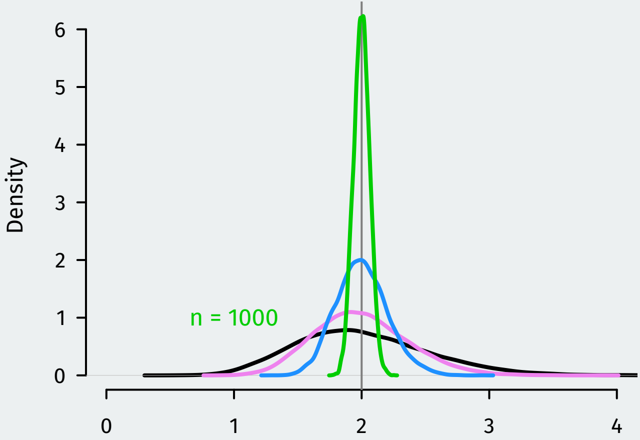
- Distribution of $\overline{X}_{30}$

LLN in action



- Distribution of $\overline{X}_{100}$

LLN in action



- Distribution of $\overline{X}_{1000}$

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- Thus, by LLN and CMT:
    - $\left(\overline{X}_n\right)^2 \xrightarrow{p} \mu^2$
    - $\log(\overline{X}_n) \xrightarrow{p} \log(\mu)$

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    - Not consistent: $\hat{\theta}_n^f$ is constant in n so its distribution never collapses.

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- Again, need to analyze when n is large.

Convergence in Distribution

Definition

Let $Z_1, Z_2, ...,$ be a sequence of r.v.s, and for $n = 1, 2, \ldots$ let $G_n(u)$ be the c.d.f. of Z_n. Then it is said that $Z_1, Z_2, \ldots$ **converges in distribution** to r.v. W with c.d.f. $G_W(u)$ if

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- We use c.d.f.s here to avoid messy details with discrete vs continuous.
- If $X_n \overset{p}{\to} X$, then $X_n \overset{d}{\to} X$

Central Limit Theorem

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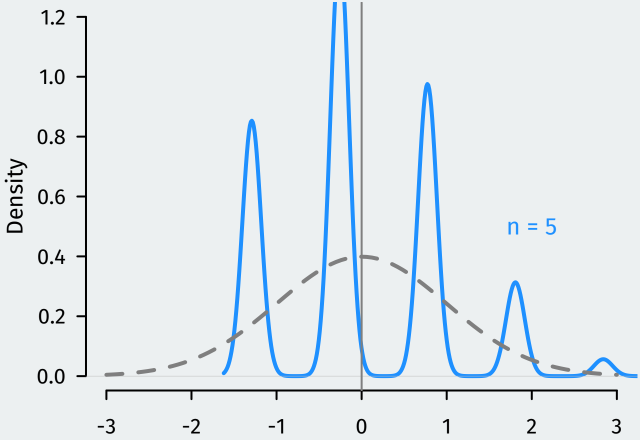
- $\rightsquigarrow$ easy approximations to probability statements about $\overline{X}_n$ when n is big!

CLT by simulation in R

```r
set.seed(02138)
nsims <- 10000
holder2 <- matrix(NA, nrow = nsims, ncol = 6)
for (i in 1:nsims) {
  s5 <- rbinom(n = 5, size = 1, prob = 0.25)
  s15 <- rbinom(n = 15, size = 1, prob = 0.25)
  s30 <- rbinom(n = 30, size = 1, prob = 0.25)
  s100 <- rbinom(n = 100, size = 1, prob = 0.25)
  s1000 <- rbinom(n = 1000, size = 1, prob = 0.25)
  s10000 <- rbinom(n = 10000, size = 1, prob = 0.25)

  holder2[i,1] <- mean(s5)
  holder2[i,2] <- mean(s15)
  holder2[i,3] <- mean(s30)
  holder2[i,4] <- mean(s100)
  holder2[i,5] <- mean(s1000)
  holder2[i,6] <- mean(s10000)
}
```

CLT in action



- Distribution of $\frac{\overline{X}_5 - \mu}{\sigma/\sqrt{5}}$

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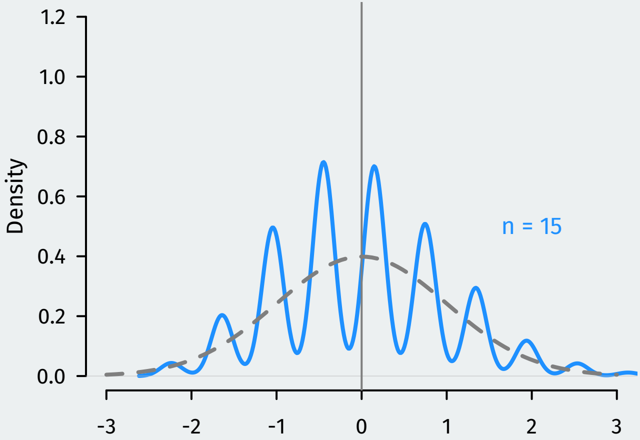

- Distribution of $\frac{\overline{X}_{15}-\mu}{\sigma/\sqrt{15}}$

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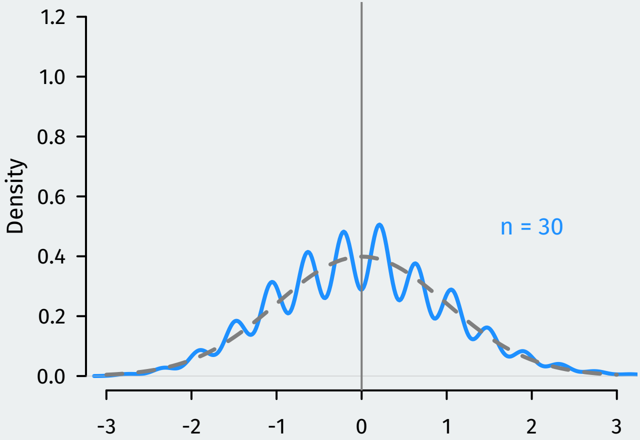

- Distribution of $\frac{\overline{X}_{30} - \mu}{\sigma / \sqrt{30}}$

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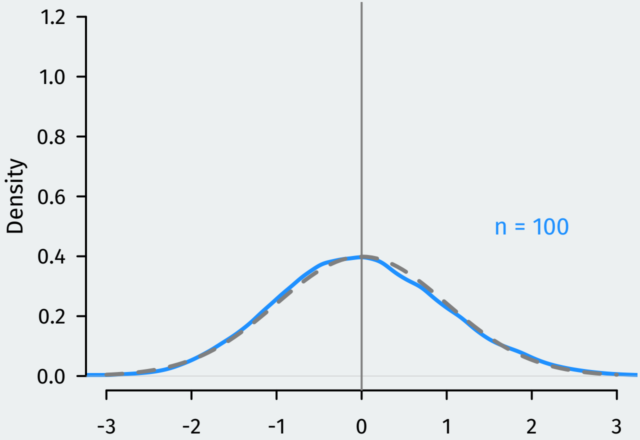
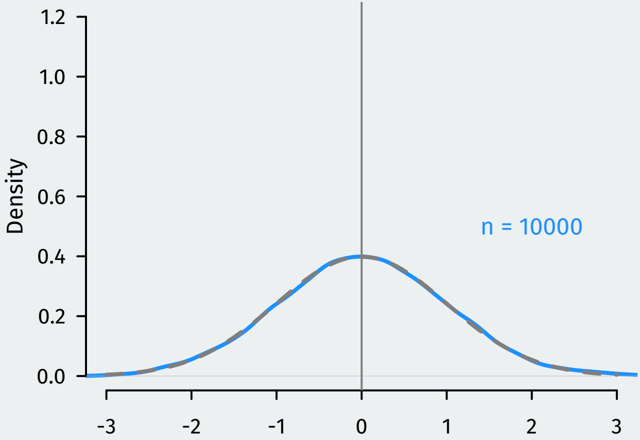


- Distribution of $\frac{\overline{X}_{100} - \mu}{\sigma / \sqrt{100}}$

CLT in action



- Distribution of $\frac{\overline{X}_{10000} - \mu}{\sigma/\sqrt{10000}}$

Transformations

- Continuous mapping theorem: for continuous g, we have

$$Z_n \xrightarrow{d} Z \qquad \Longrightarrow \qquad g(Z_n) \xrightarrow{d} g(Z).$$

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    3. X_n / Y_n converges in distribution to X/c if $c \neq 0$
- Extremely useful when trying to figure out what the large-sample distribution of an estimator is.

Delta method

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If $\sqrt{n}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} \mathcal{N}(0, V)$ and $h(u)$ is continuously differentiable in a neighborhood around θ, then as $n \to \infty$,

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- Why $h()$ continuously differentiable?

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  - So $h(\hat{\theta}_n) - h(\theta) \approx h'(\theta)\left(\hat{\theta}_n - \theta\right)$

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- But we don't know V_θ?! Estimate it!

$$\widehat{V}_\theta = \frac{1}{n}\sum_{i=1}^{n}\left(g(X_i) - \hat{\theta}_n\right)^2$$

Variance estimation with plug-in estimators

- Setting: $X_1, \ldots, X_n$ i.i.d. with quantity of interest $\theta = \mathbb{E}[g(X_i)]$
- Analogy/plug-in estimator: $\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} g(X_i)$, by CLT:

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} \mathcal{N}(0, V_\theta)$$

where $V_\theta = \mathbb{V}[g(X_i)] = \mathbb{E}[(g(X_i) - \theta)^2]$.

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- We can show that $\widehat{V_\theta} \xrightarrow{p} V_\theta$ and so by Slutsky:

$$\frac{\sqrt{n}\left(\hat{\theta}_n - \theta\right)}{\sqrt{\widehat{V_\theta}}} \xrightarrow{d} \frac{\mathcal{N}(0, V_\theta)}{\sqrt{V_\theta}} \sim \mathcal{N}(0, 1)$$

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  - Basically: multivariate CLT holds if each r.v. in the vector has finite variance.
- Very common for when we're estimating multiple parameters $\boldsymbol{\theta}$ with $\widehat{\boldsymbol{\theta}}_n$

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Bounded in probability

Definition

A random sequence Z_n is **bounded in probability**, written $Z_n = O_p(1)$ ("big-oh-p-one") for all $\delta > 0$ there exists a M_δ and n_δ, such that for $n \geq n_\delta$,

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