11. Confidence Intervals

Spring 2021

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Gov 2002 (Harvard)

Interval estimation - what and why?

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- We can use the distribution of estimators to derive these intervals.

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- Important: interval is the random quantity, not the parameter.

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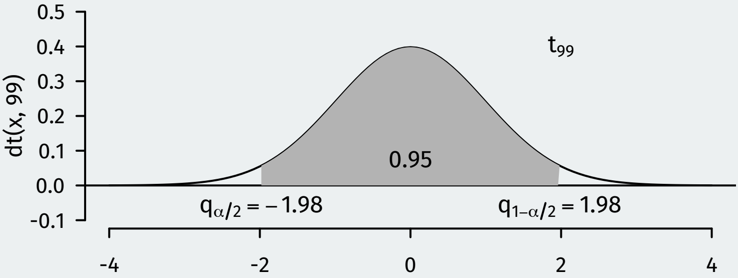
- If $q_{1-\alpha/2}$ is the $1 - \alpha/2$ quantile of the t_{n-1}, we have

$$\mathbb{P}\left(-q_{1-\alpha/2} \leq \frac{\overline{X}_n - \mu}{s/\sqrt{n}} \leq q_{1-\alpha/2}\right) = 1 - \alpha$$

Finding the critical values



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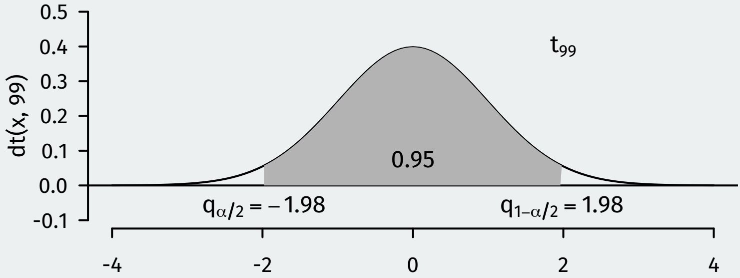

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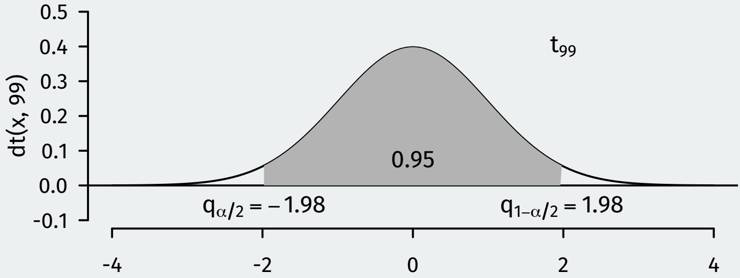

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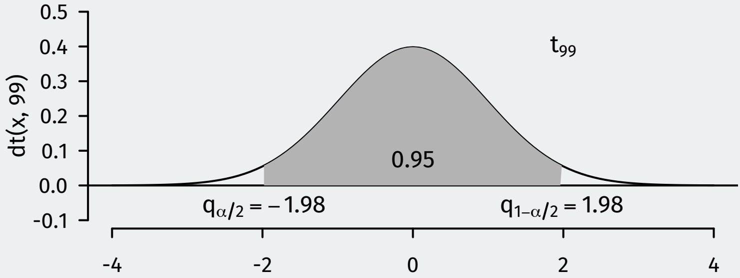

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  - If $G(t)$ is the c.d.f. of T, then we have $q_{1-\alpha/2} = G^{-1}(1 - \alpha/2)$

Deriving the interval

- Let's work backwards to derive the confidence interval:

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- Bounds are random! Not μ!

Asymptotic confidence intervals

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- Again, $z_{1-\alpha/2} = \Phi^{-1}(1-\alpha/2)$ (`qnorm` in R)

CI for social pressure effect

TABLE 2. Effects of Four Mail Treatments on Voter Turnout in the August 2006 Primary Election

	Experimental Group				
	Control	Civic Duty	Hawthorne	Self	Neighbors
Percentage Voting	29.7%	31.5%	32.2%	34.5%	37.8%
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```
neigh_var <- var(social$voted[social$treatment == "Neighbors"])
neigh_n <- 38201
civic_var <- var(social$voted[social$treatment == "Civic Duty"])
civic_n <- 38218

se_diff <- sqrt(neigh_var/neigh_n + civic_var/civic_n)

## c(lower, upper)
c((0.378 - 0.315) - 1.96 * se_diff, (0.378 - 0.315) + 1.96 * se_diff)
```

```
## [1] 0.0563 0.0697
```

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- Correct interpretation: **across 95% of random samples, the constructed confidence interval will contain the true value.**

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```r
sims<- 10000
cover <- rep(0, times = sims)
low.bound <- up.bound <- rep(NA, times = sims)
for(i in 1:sims){
    draws <- rnorm(500, mean = 1, sd = sqrt(10))
    low.bound[i] <- mean(draws) - sd(draws) / sqrt(500) * 1.96
    up.bound[i] <- mean(draws) + sd(draws) / sqrt(500) * 1.96
    if (low.bound[i] < 1 & up.bound[i] > 1) {
        cover[i] <- 1
    }
}
mean(cover)
```

```
## [1] 0.95
```

Plotting the CIs

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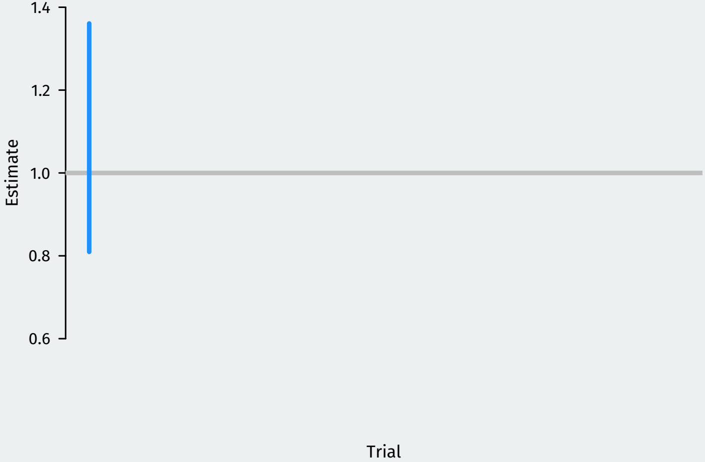

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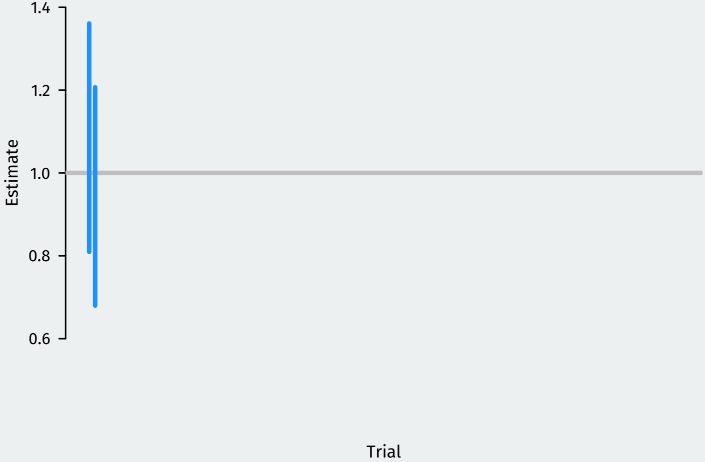

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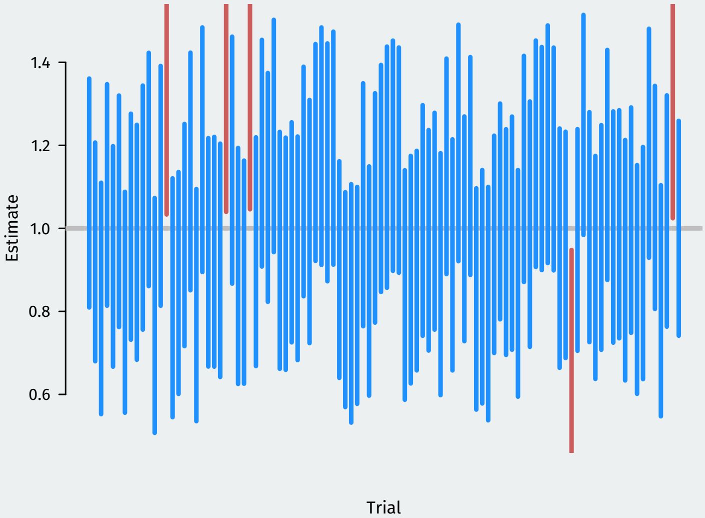

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$$T = \frac{\overline{X}_n - \mu_0}{s/\sqrt{n}}$$

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