12. Conditional Expectation

Spring 2021

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Gov 2002 (Harvard)

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- At its core: how the average of one variable varies with others.

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Definition

The **conditional expectation** of Y conditional on $\mathbf{X} = \mathbf{x}$ is:

$$\mu(\mathbf{x}) = \mathbb{E}[Y \mid \mathbf{X} = \mathbf{x}] = \begin{cases} \sum_y y \; \mathbb{P}(Y = y \mid \mathbf{X} = \mathbf{x}) & \text{discrete } Y \\ \int_{-\infty}^{\infty} y \; f_{Y|\mathbf{X}}(y \mid \mathbf{x}) dy & \text{continuous } Y \end{cases}$$

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- Viewed as a function of x, it is the **conditional expectation function (CEF)**
    - How does the average value of Y change given different levels of $\mathbf{X}$?

Conditional expectation example

	Support Gay Marriage $Y = 1$	Oppose Gay Marriage $Y = 0$
Female $X = 1$	0.30	0.21
Male $X = 0$	0.22	0.27

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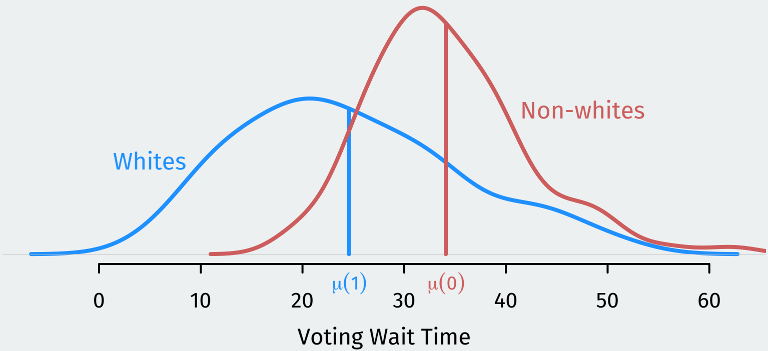
- Example:
  - Y_i is the time respondent i waited in line to vote.
  - $X_i = 1$ for whites, $X_i = 0$ for non-whites.
- Then the mean in each group is just a conditional expectation:

$$\mu(\text{white}) = E[Y_i | X_i = \text{white}]$$
$$\mu(\text{non-white}) = E[Y_i | X_i = \text{non-white}]$$

Why is the CEF useful?



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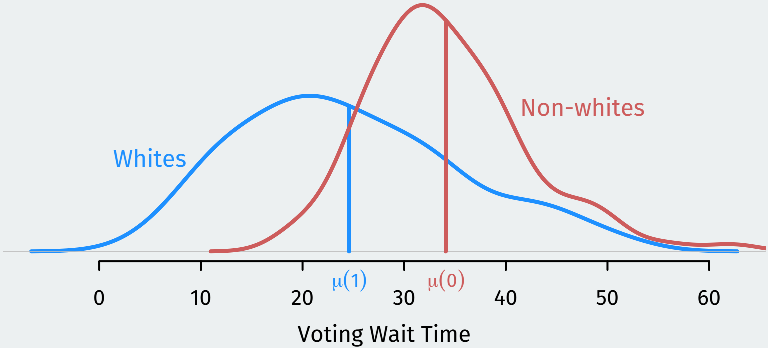

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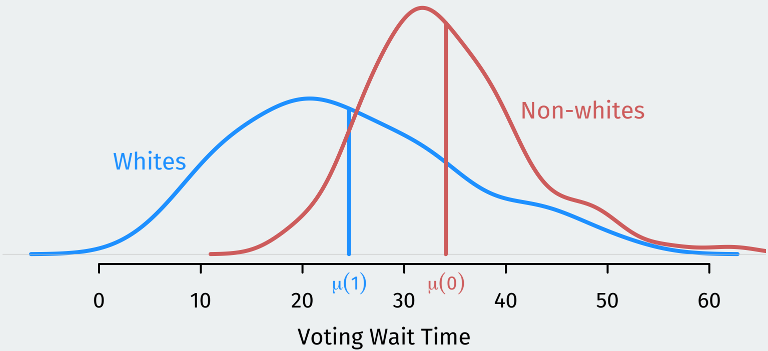

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- Indicates a relationship **in the population** between race and wait times.

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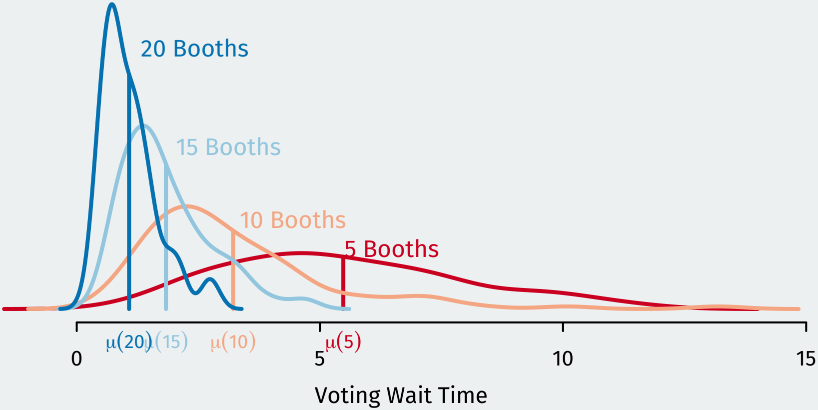

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- Ex: average difference in wait times between white and non-white citizens **of the same gender**:

$$\mu(\text{white}, \text{man}) - \mu(\text{non-white}, \text{man})$$

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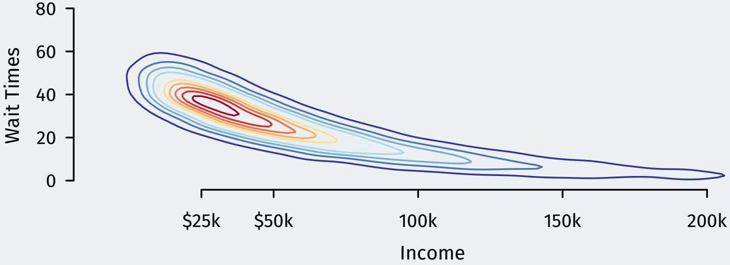
- These are **unknown functions in the population**! This is going to make producing an estimator $\hat{\mu}(x)$ very difficult!

Wait times and income



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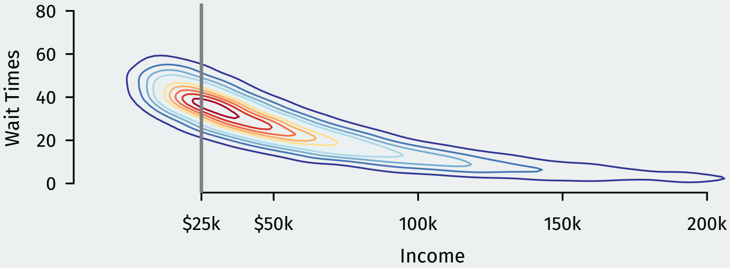

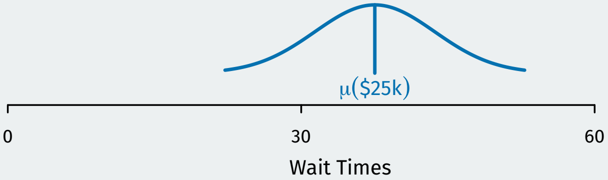

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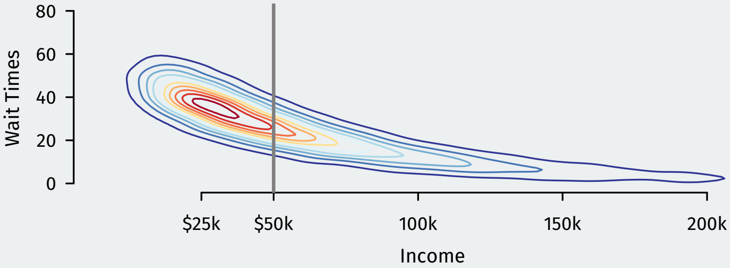

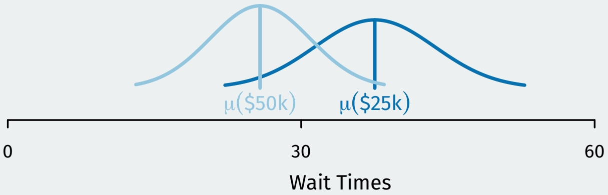

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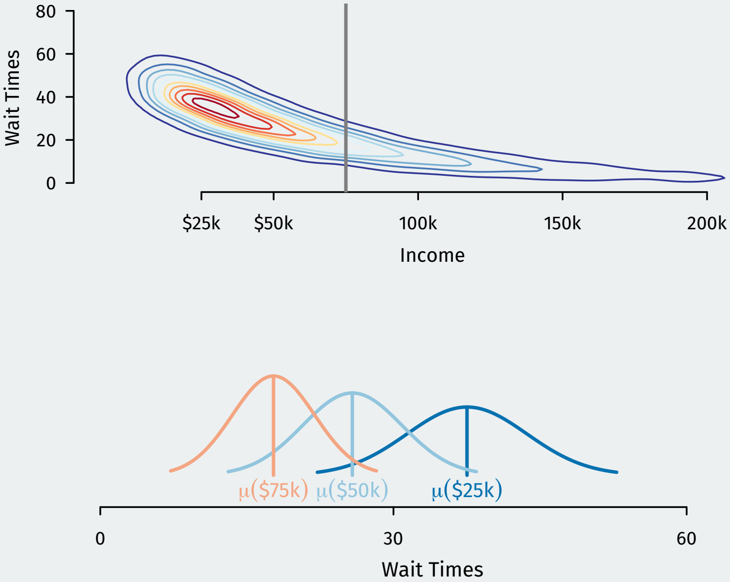

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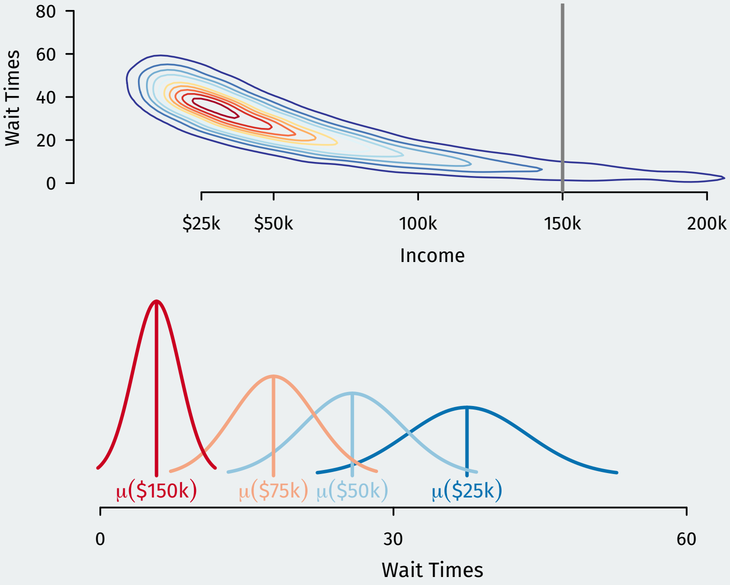

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- Has an expectation, $\mathbb{E}[\mathbb{E}[Y \mid X]]$, and a variance, $\mathbb{V}[\mathbb{E}[Y \mid X]]$.

Law of iterated expectations

Simple Law of Iterated Expectations

If $\mathbb{E}|Y| < \infty$, for any random vector $\mathbf{X}$, $\mathbb{E}\left\{\mathbb{E}[Y \mid \mathbf{X}]\right\} = E[Y]$.

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- "Averaging" over what is not constant ($\mathbf{X}_2$).

Example: law of iterated expectations

	Support Gay Marriage $Y = 1$	Oppose Gay Marriage $Y = 0$	Marginal
Female $X = 1$	0.30	0.21	0.51
Male $X = 0$	0.22	0.27	0.49
Marginal	0.52	0.48	

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- These properties are definitional, not assumptions.

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- If $E[Y^2] < \infty$, then for any predictor $g(\mathbf{X})$,

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Definition

The **conditional variance** of a Y given $\mathbf{X} =$ is defined as:

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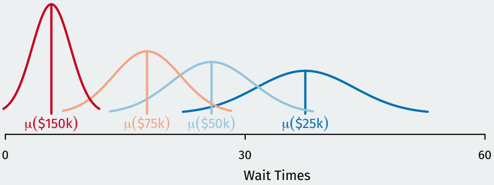
- Can re-express in the usual way:

$$\mathbb{V}[Y \mid \mathbf{X} = \mathbf{x}] = \mathbb{E}\left[Y^2 \mid \mathbf{X} = \mathbf{x}\right] - \left(\mathbb{E}[Y \mid \mathbf{X} = \mathbf{x}]\right)^2$$

Skedasticity



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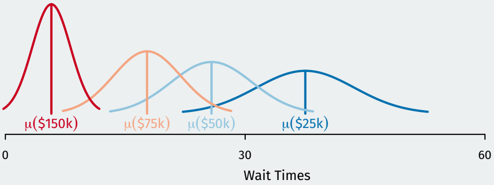

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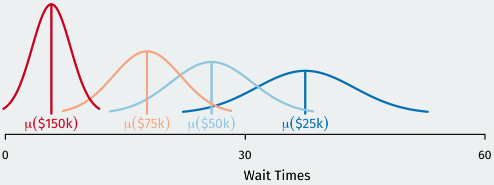

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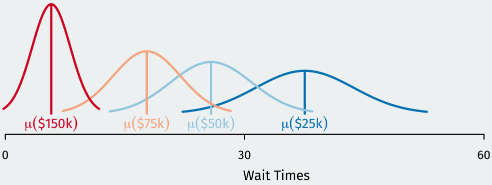

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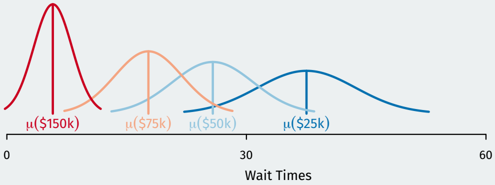

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- Default assumption should be the less restrictive one: heteroskedastic

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