13. Linear Model

Spring 2021

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Gov 2002 (Harvard)

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- Now: focusing on when the CEF is (and isn't) linear.
- Linear model is ubiquitous but poorly understood. Lots of subtlety here.

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    - How do we decide what form $\mu(x)$ should take?

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- More generally for any discrete X_i:

$$\hat{\mu}(x) = \frac{\sum_{i=1}^{N} Y_i \mathbb{I}(X_i = x)}{\sum_{i=1}^{N} \mathbb{I}(X_i = x)}$$

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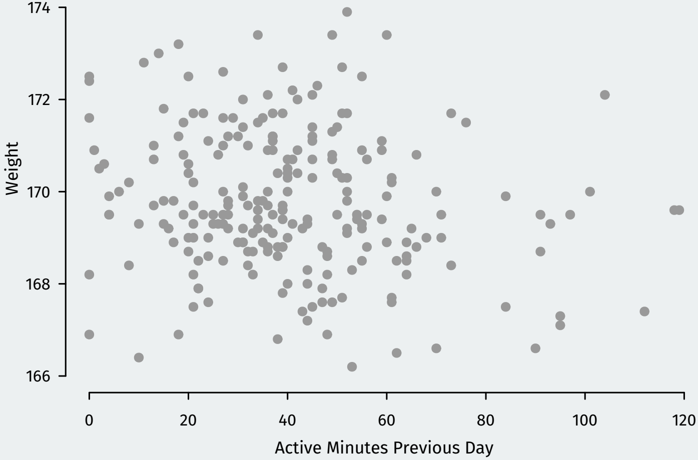
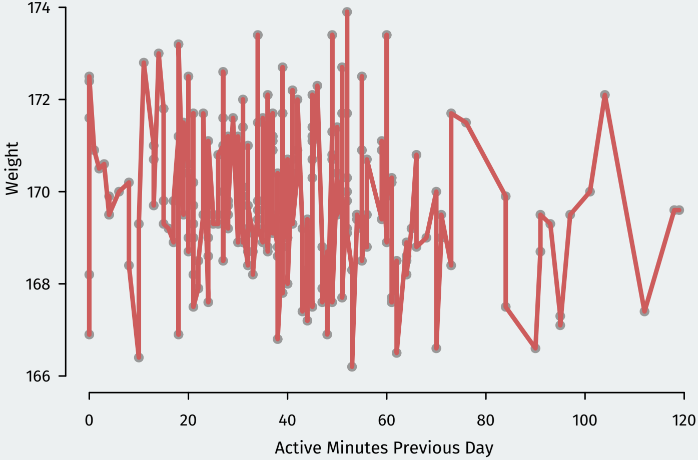
  - Personal data science: I wear an activity tracker and have a smart scale.
  - Relationship between my weight and active minutes in the previous day.

Continuous covariate example



Continuous covariate CEF: interpolation



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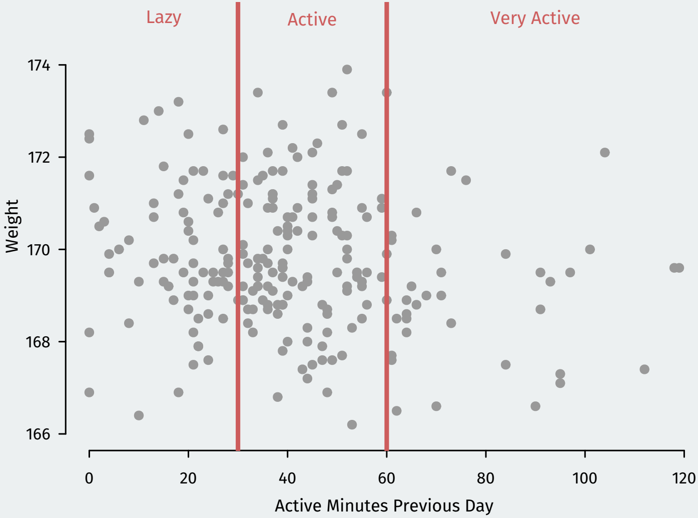

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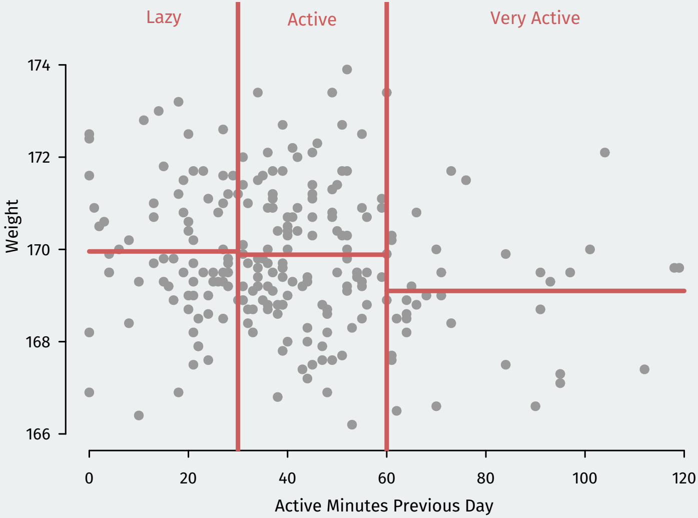

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- **Intercept**, β_0: the condition expectation of Y_i when $X_i = 0$
- **Slope**, β_1: change in the CEF of Y_i given a one-unit change in X_i

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- Put another way: average partial effects are constant $\frac{\partial \mu(x)}{\partial x} = \beta_1$

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  - Average partial effect of X_1 depends on X_2: $\partial\mu(x_1, x_2)/\partial x_1 = \beta_1 + x_2\beta_3$

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- $>$ 2 categories: dummies for all but category and everything is linear.

Linear CEF with multiple binary covariates

- What if we have two binary covariates, X_1 (race) and X_2 (1 urban/0 rural):

$$\mu(x_1, x_2) = \begin{cases} \mu_{00} & \text{if } x_1 = 0 \text{ and } x_2 = 0 \text{ (POC, rural)} \\ \mu_{10} & \text{if } x_1 = 1 \text{ and } x_2 = 0 \text{ (white, rural)} \\ \mu_{01} & \text{if } x_1 = 0 \text{ and } x_2 = 1 \text{ (POC, urban)} \\ \mu_{11} & \text{if } x_1 = 1 \text{ and } x_2 = 1 \text{ (white, urban)} \end{cases}$$

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  - $\beta_2 = \mu_{01} - \mu_{00}$: diff. in means for urban POC vs rural POC.

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    - $\beta_3 = (\mu_{11} - \mu_{01}) - (\mu_{10} - \mu_{00})$: diff. in urban racial diff. vs rural racial diff.

Linear CEF with multiple binary covariates

- What if we have two binary covariates, X_1 (race) and X_2 (1 urban/0 rural):

$$\mu(x_1, x_2) = \begin{cases} \mu_{00} & \text{if } x_1 = 0 \text{ and } x_2 = 0 \text{ (POC, rural)} \\ \mu_{10} & \text{if } x_1 = 1 \text{ and } x_2 = 0 \text{ (white, rural)} \\ \mu_{01} & \text{if } x_1 = 0 \text{ and } x_2 = 1 \text{ (POC, urban)} \\ \mu_{11} & \text{if } x_1 = 1 \text{ and } x_2 = 1 \text{ (white, urban)} \end{cases}$$

- Can rewrite this without assumptions as a linear CEF with interaction:

$$\mu(x_1, x_2) = \beta_0 + x_1\beta_1 + x_2\beta_2 + x_1x_2\beta_3$$

- Interpretations:
    - $\beta_0 = \mu_{00}$: average wait times for rural POC.
    - $\beta_1 = \mu_{10} - \mu_{00}$: diff. in means for rural whites vs rural POC.
    - $\beta_2 = \mu_{01} - \mu_{00}$: diff. in means for urban POC vs rural POC.
    - $\beta_3 = (\mu_{11} - \mu_{01}) - (\mu_{10} - \mu_{00})$: diff. in urban racial diff. vs rural racial diff.
- Generalizes to p binary variables if **all interactions included (saturated)**

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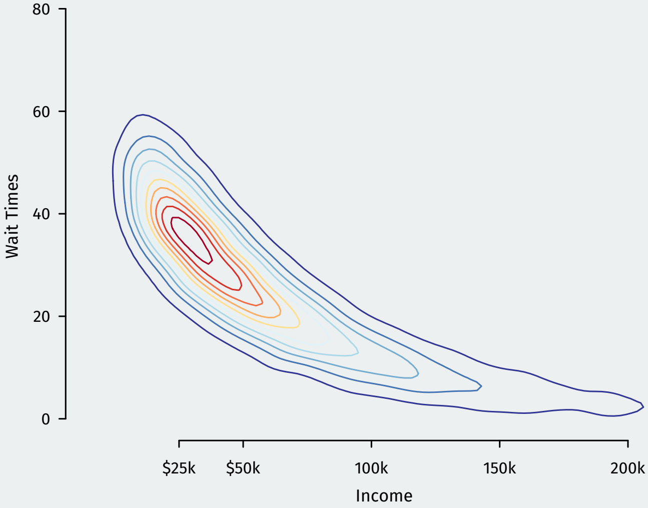

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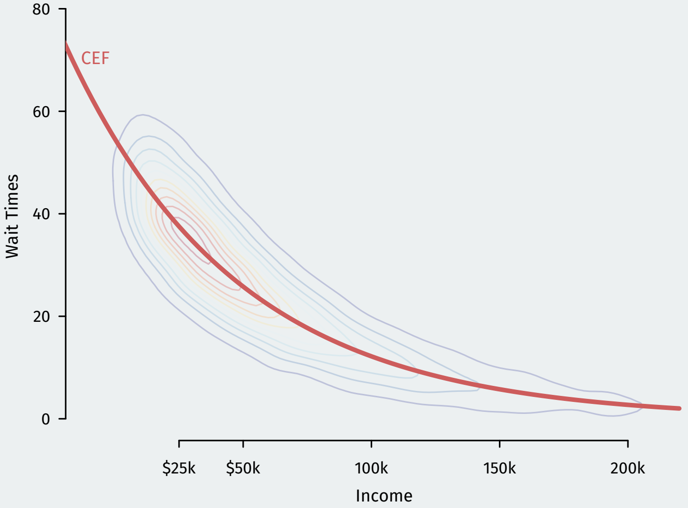

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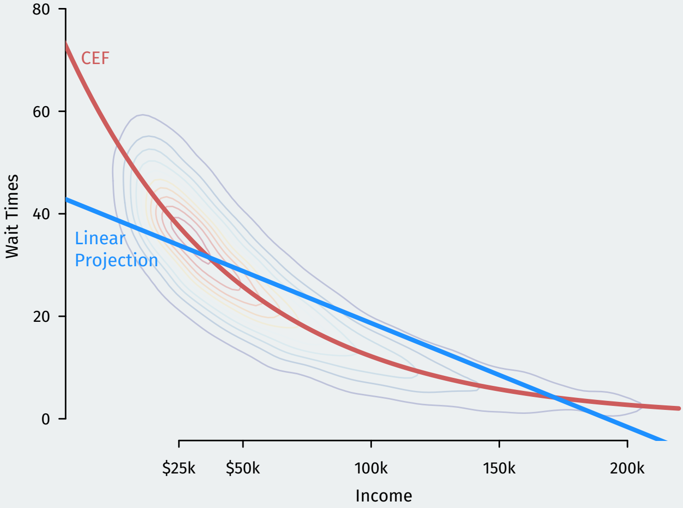

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    3. $\mathbf{Q_{XX}} = \mathbb{E}[\mathbf{XX}']$ is positive definite (columns of **X** are linearly independent)

Linear Projection

- How to find $\boldsymbol{\beta}$? Minimize squared prediction error!

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Regression coefficients

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- Solution for $\boldsymbol{\beta}$ more interpretable here:

$$\boldsymbol{\beta} = \mathbb{V}[\mathbf{X}]^{-1}\text{Cov}(\mathbf{X}, Y), \qquad \beta_0 = \mu_Y - \boldsymbol{\mu}_{\mathbf{X}}'\boldsymbol{\beta}$$

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- Also holds if we get residuals from projection of Y on $\mathbf{Z}$: $V = Y - \mathbb{L}(Y \mid \mathbf{Z})$.

$$\mathbb{L}(V \mid \mathbf{R}) = \mathbf{R}'\boldsymbol{\beta}$$

Omitted variable bias

- Consider two projections/regressions with and without some Z:

$$\mathbb{L}[Y \mid \mathbf{X}, \mathbf{Z}] = \mathbf{X}'\boldsymbol{\beta} + Z\gamma, \qquad \mathbb{L}[Y \mid \mathbf{X}] = \mathbf{X}'\boldsymbol{\delta}$$

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- $\boldsymbol{\beta} - \boldsymbol{\delta} = \boldsymbol{\pi}\gamma$ is the “bias” but this is misleading.

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